

The quantum state vector in phase space and Gabor's windowed Fourier transform

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Abstract

Representations of quantum state vectors by complex phase space amplitudes, complementing the description of the density operator by the Wigner function, have been defined by applying the Weyl-Wigner transform to dyadic operators, linear in the state vector and anti-linear in a fixed 'window state vector'. Here aspects of this construction are explored, and a connection is established with Gabor's 'windowed Fourier transform'. The amplitudes that arise for simple quantum states from various choices of window are presented as illustrations. Generalized Bargmann representations of the state vector appear as special cases, associated with Gaussian windows. For every choice of window, amplitudes lie in a corresponding linear subspace of square-integrable functions on phase space. A generalized Born interpretation of amplitudes is described, with both the Wigner function and a generalized Husimi function appearing as quantities linear in an amplitude and anti-linear in its complex conjugate. Schrödinger's time-dependent and time-independent equations are represented on phase space amplitudes, and their solutions described in simple cases.

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1 Introduction

The phase space formulation of quantum mechanics [1, 2, 3, 4] now plays a central role in theoretical quantum optics [5] and quantum tomography [6], and it has also become an important vehicle for investigations of fundamental questions such as the nature of quantization [7, 8] and the quantum-classical interface [9, 10, 11]. New features continue to be revealed by ongoing investigations of the underlying structures [12, 13, 14].

The overwhelming majority of the very large number of studies in this field have focussed on the Wigner function W , the real-valued quasiprobability density on phase space Γ which is the representative of the density operator $\hat{\rho}$ under the action of the Weyl-Wigner transform \mathcal{W} . However, in the last two decades a small number of studies [15, 16, 17, 18, 19, 20, 21] have gradually made clear how the state vector $|\psi\rangle$, which is after all a more fundamental object than $\hat{\rho}$, can also be depicted in the phase space picture, as a complex-valued amplitude Ψ on Γ . The recent work of Smith [21] in particular shows that this is achieved most simply by applying \mathcal{W} to a suitable multiple of the dyadic operator $|\psi\rangle\langle\varphi_0|$, where $|\varphi_0\rangle$ is an arbitrary normalized state vector that, once chosen, is held fixed while $|\psi\rangle$ ranges over all states of the given quantum system. From this definition it follows that Ψ is a multiple of the two-state Wigner function [22] $W_{\psi\varphi_0}$, which is thereby given an important role in the phase space formulation.

Several questions concerning this important extension of the phase space formulation suggest themselves:

- In what sense is $|\psi\rangle\langle\varphi_0|$, and hence its image Ψ , equivalent to $|\psi\rangle$ in quantum mechanical calculations?
- What relationships do the Wigner function W and the Husimi function [23] have with the phase space amplitude Ψ ?
- What form does the amplitude Ψ take for special system states $|\psi\rangle$ such as coherent states, or eigenstates of position or momentum, for different choices of $|\varphi_0\rangle$?
- For a given $|\psi\rangle$, how does the choice of $|\varphi_0\rangle$ influence the structure of Ψ and its place in the set of functions on Γ ?
- What forms do the time-dependent and time-independent Schrödinger equations take on phase space amplitudes, and how do their solutions

look in simple cases, for various choices of $|\varphi_0\rangle$?

- What is the nature of the mapping from $|\psi\rangle$ to Ψ , when viewed as a transform of the coordinate space wavefunction ψ , and how does it relate to other, well-known transforms in the literature?

These are the questions that we attempt to address in what follows. With regard to the last question, we shall see that the transform in question is closely related to Gabor’s ‘windowed Fourier transform’ [24], widely used in the signal processing literature [25, 26] and also called there the ‘short-time Fourier transform’. Accordingly we shall refer to the fixed state $|\varphi_0\rangle$ that appears in the definition of phase space amplitudes Ψ as the window vector, window state, or simply ‘the window’ in what follows. Other names for closely related objects in the literature are ‘probe functions’ [15] and ‘drone states’ or ‘fiducial states’ [21]. We shall see also that the choice of an oscillator ground state as window gives a Ψ that is, up to a factor independent of the state $|\psi\rangle$, the well-known Bargmann wavefunction [27], which is thereby seen as a precursor to the more recent efforts [15, 16, 17, 18, 19, 20, 21] to represent the state vector in phase space.

For simplicity of presentation in what follows, we consider mainly one linear degree of freedom, ignore spin degrees of freedom, and treat all variables as dimensionless, setting Planck’s constant \hbar equal to 1. We use hats to label operators on the usual complex Hilbert space \mathcal{H} of quantum mechanics. Variables without hats are defined on Γ , unless otherwise specified.

2 Definition of phase space amplitudes

Our starting point is the observation that pure state vectors $|\psi\rangle$ can be replaced in all quantum mechanical calculations, without any loss of generality, by the corresponding dyadic operators $|\psi\rangle\langle\varphi_0|$, for any *fixed* vector $|\varphi_0\rangle$ of unit length, $\langle\varphi_0|\varphi_0\rangle = 1$. For example, we can superpose dyadic operators,

$$\alpha|\psi_1\rangle + \beta|\psi_2\rangle \longleftrightarrow \alpha|\psi_1\rangle\langle\varphi_0| + \beta|\psi_2\rangle\langle\varphi_0|; \quad (1)$$

we can evolve them in time using Schrödinger’s equation,

$$i\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \longleftrightarrow i\partial_t|\psi(t)\rangle\langle\varphi_0| = \hat{H}|\psi(t)\rangle\langle\varphi_0|; \quad (2)$$

we can use them to calculate corresponding pure-state density operators,

$$\hat{\rho} = |\psi\rangle\langle\psi| = |\psi\rangle\langle\varphi_0|(|\psi\rangle\langle\varphi_0|)^\dagger; \quad (3)$$

we can use them to calculate expectation values,

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle = \text{Tr} \left((|\psi\rangle\langle\varphi_0|)^\dagger \hat{A} |\psi\rangle\langle\varphi_0| \right); \quad (4)$$

and, most important, we can use them to calculate transition amplitudes,

$$\langle\psi_1|\psi_2\rangle = \text{Tr} \left((|\psi_1\rangle\langle\varphi_0|)^\dagger |\psi_2\rangle\langle\varphi_0| \right), \quad (5)$$

and not just transition *probabilities* $|\langle\psi_1|\psi_2\rangle|^2$. Transition probabilities, but not transition amplitudes, can readily be determined from the density matrix or, equivalently, from the Wigner function.

It is now a simple matter [21] to combine two ideas — the notion of replacing state vectors $|\psi\rangle$ by dyadic operators $|\psi\rangle\langle\varphi_0|$, and the notion of mapping operators into functions on phase space using the Weyl-Wigner transform — in order to define phase space amplitudes $\Psi(q, p)$. We simply set

$$\Psi(q, p) = \frac{1}{\sqrt{2\pi}} \mathcal{W}(|\psi\rangle\langle\varphi_0|)(q, p) \equiv \frac{1}{\sqrt{2\pi}} \text{Tr} \left(|\psi\rangle\langle\varphi_0| \hat{\Delta}(q, p) \right). \quad (6)$$

Here $\hat{\Delta}(q, p)$ is the Weyl-Wigner-Stratonovich kernel operator [28, 29, 30, 31, 32, 33, 34, 35], given in terms of q, p and corresponding (dimensionless) canonical operators \hat{q}, \hat{p} , by

$$\hat{\Delta}(q, p) = 2 e^{2i(p\hat{q} - q\hat{p})} \hat{\Pi}, \quad (7)$$

where $\hat{\Pi}$ is the parity operator, satisfying

$$\hat{q} \hat{\Pi} = -\hat{\Pi} \hat{q}, \quad \hat{p} \hat{\Pi} = -\hat{\Pi} \hat{p}, \quad \hat{\Pi}^2 = 1. \quad (8)$$

We recall at this point that the star product [36, 37, 38] of phase space functions is defined through the Weyl-Wigner transform by

$$\mathcal{W}(\hat{A}) = A, \quad \mathcal{W}(\hat{B}) = B \Rightarrow \mathcal{W}(\hat{A}\hat{B}) = A \star B, \quad (9)$$

leading to

$$\begin{aligned}
(A \star B)(q, p) &= A(q, p) e^{iJ/2} B(q, p), \quad (J = \overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q) \\
&= e^{i(\partial_q \partial_{p'} - \partial_p \partial_{q'})/2} A(q, p) B(q', p')|_{(q', p')=(q, p)}. \quad (10)
\end{aligned}$$

The expressions involving differential operators are well-defined if A and B are polynomial functions. In more general cases, they define asymptotic expansions,

$$A(q, p) \star B(q, p) = A(q, p) B(q, p) + \frac{1}{2} i A(q, p) J B(q, p) + O(2), \quad (11)$$

provided A and B represent observables that are asymptotically regular at $\hbar = 0$ [39]. In (11), $O(2)$ denotes terms of second order in Planck's constant (here implicit).

A string of results now follows from (6) and (9-11):

1. *Complex phase space amplitudes:* For each choice of window vector $|\varphi_0\rangle$, a (distinct) set of amplitudes Ψ is defined by (6), corresponding to the set of all state vectors $|\psi\rangle$. Each Ψ therefore carries implicitly a label φ_0 , which we suppress. For any choice of $|\varphi_0\rangle$, the amplitudes Ψ are complex-valued in general, and carry the same phases as the corresponding state vectors $|\psi\rangle$, up to a constant phase shift determined by $|\varphi_0\rangle$.
2. *Expectation values:* Corresponding to (4), expectation values can be expressed in the form

$$\begin{aligned}
\langle \hat{A} \rangle &= \int \overline{\Psi(q, p)} \star A(q, p) \star \Psi(q, p) d\Gamma \\
&= \int \left\{ \Psi(q, p) \star \overline{\Psi(q, p)} \right\} A(q, p) d\Gamma \\
&= \int \overline{\Psi(q, p)} \{ A(q, p) \star \Psi(q, p) \} d\Gamma. \quad (12)
\end{aligned}$$

Here we have used the result [40, 41]

$$\text{Tr}(\widehat{A}\widehat{B}) \longrightarrow \int A(q, p) \star B(q, p) d\Gamma = \int A(q, p) B(q, p) d\Gamma, \quad (13)$$

for suitably smooth functions A and B .

3. *Transition probabilities:* Corresponding to (5), transition amplitudes and subsequently transition probabilities can be calculated from

$$\langle \psi_1 | \psi_2 \rangle = \int \overline{\Psi_1(q, p)} \Psi_2(q, p) d\Gamma. \quad (14)$$

In particular,

$$\langle \psi | \psi \rangle = \int \overline{\Psi(q, p)} \Psi(q, p) d\Gamma = 1 \quad (15)$$

which suggests that $|\Psi(q, p)|^2$, like $W(q, p)$, can be regarded as a quasi-probability distribution over Γ in its own right. We shall see below that in fact $|\Psi(q/2, p/2)/2|^2$ is a generalized Husimi distribution.

4. *Generalized Born interpretation:* Corresponding to (3), we have

$$W(q, p) = \Psi(q, p) \star \overline{\Psi(q, p)}, \quad (16)$$

which, bearing (12) in mind, we refer to as the Born interpretation of Ψ , by analogy with the relation between the wavefunction and probability density in configuration space. Then (11) suggests the expansion

$$W(q, p) = \Psi(q, p) \overline{\Psi(q, p)} + \frac{1}{2} i \Psi(q, p) J \overline{\Psi(q, p)} + \dots, \quad (17)$$

which in turn suggests a non-negative approximation to the Wigner function,

$$W(q, p) \approx |\Psi(q, p)|^2 \geq 0. \quad (18)$$

However, (17) and (18) must be treated with caution because Ψ and W may not be asymptotically regular at $\hbar = 0$ [39, 42]. Comparisons of $|\Psi|^2$ with W in examples that follow illustrate the difficulty — see for example (73) and (74).

5. *Superposition property:* For each choice of $|\varphi_0\rangle$, the mapping from $|\psi\rangle$ to Ψ is linear and phase space amplitudes can be superposed,

$$|\psi_{12}\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \longleftrightarrow \Psi_{12}(q, p) = c_1\Psi_1(q, p) + c_2\Psi_2(q, p), \quad (19)$$

preserving the phase relations between state vectors.

Note from (16) that if W_1 , W_2 and W_{12} are the Wigner functions corresponding to the states and phase space amplitudes in (19), then

$$\begin{aligned} W_{12} &= (c_1\Psi_1 + c_2\Psi_2) \star \overline{(c_1\Psi_1 + c_2\Psi_2)} \\ &= |c_1|^2 W_1 + |c_2|^2 W_2 \\ &\quad + c_1 \overline{c_2} \Psi_1 \star \overline{\Psi_2} + c_2 \overline{c_1} \Psi_2 \star \overline{\Psi_1}. \end{aligned} \quad (20)$$

In contrast, it is possible [43] but not straightforward to express W_{12} , for example, directly in terms of W_1 and W_2 .

Similarly for the generalized Husimi distribution, we have

$$\begin{aligned} |\Psi_{12}|^2 &= (c_1\Psi_1 + c_2\Psi_2) \overline{(c_1\Psi_1 + c_2\Psi_2)} \\ &= |c_1|^2 |\Psi_1|^2 + |c_2|^2 |\Psi_2|^2 \\ &\quad + c_1 \overline{c_2} \Psi_1 \overline{\Psi_2} + c_2 \overline{c_1} \Psi_2 \overline{\Psi_1}. \end{aligned} \quad (21)$$

6. *Normalization and subspace of amplitudes:* The results (14) and (15) show that, for whatever choice of $|\varphi_0\rangle$, the complex amplitude Ψ belongs to the Hilbert space of square-integrable functions on phase space, and is normalized when $|\psi\rangle$ is normalized. In fact all such amplitudes for a given window state lie in a proper closed subspace \mathcal{S}_{φ_0} of that Hilbert space, characterized by the property that

$$\Psi(q, p) \star W_{\varphi_0}(q, p) = \Psi(q, p)/2\pi, \quad (22)$$

where W_{φ_0} is the Wigner function corresponding to the state $|\varphi_0\rangle$. This follows from the identity

$$|\psi\rangle\langle\varphi_0|\varphi_0\rangle\langle\varphi_0| = |\psi\rangle\langle\varphi_0| \quad (23)$$

under the action of \mathcal{W} . Note also the identity

$$W_{\psi}(q, p) \star \Psi(q, p) = \Psi(q, p)/2\pi, \quad (24)$$

which follows from

$$|\psi\rangle\langle\psi|\psi\rangle\langle\varphi_0| = |\psi\rangle\langle\varphi_0|. \quad (25)$$

It can be seen from (14) that if $\{|\psi_n\rangle\}$ is a complete orthonormal set of vectors in \mathcal{H} , then the corresponding set of phase space amplitudes $\{\Psi_n\}$ provides a complete orthonormal set in \mathcal{S}_{φ_0} under the scalar product given by the RHS of (14). Thus \mathcal{S}_{φ_0} is a Hilbert space in its own right, which for each choice of window provides an image of \mathcal{H} within the space of square-integrable functions on Γ . It is clear [44] that \mathcal{H} , which after all is commonly realized in the coordinate representation as the space of square-integrable functions on the real *line*, has uncountably many images in the space of square-integrable functions on the phase *plane*.

3 A class of integral transforms

In the coordinate representation, formula (6) takes the form

$$\begin{aligned}
\Psi(q, p) &= \frac{1}{\sqrt{2\pi}} \int \psi(q - \tfrac{1}{2}y) \overline{\varphi_0(q + \tfrac{1}{2}y)} e^{ipy} dy \\
&= \sqrt{2/\pi} \int \psi(u) \overline{\varphi_0(2q - u)} e^{2ip(q-u)} du \\
&= \sqrt{2/\pi} \int \psi(2q - u) \overline{\varphi_0(u)} e^{2ip(u-q)} du, \tag{26}
\end{aligned}$$

where ψ and φ_0 are the wavefunctions corresponding to $|\psi\rangle$ and $|\varphi_0\rangle$ respectively.

The first of these formulas shows that, apart from a normalization factor, Ψ is the ‘two-state’ or ‘two-sided’ Wigner function [3, 22] that is sometimes denoted $W_{\psi\varphi_0}(q, p)$. But as the notation Ψ suggests, we consider it now as varying with, and determined by $|\psi\rangle$, with $|\varphi_0\rangle$ held fixed once and for all. We emphasize in particular that when $|\psi(t)\rangle$ evolves in time, $\Psi(t)$ evolves also with *only the LH member* of $|\psi(t)\rangle\langle\varphi_0|$ changing (see below).

The second formula (26) shows that the transform from ψ to Ψ , obtained with any fixed choice of the normalized wavefunction φ_0 , is closely related to Gabor’s ‘windowed Fourier transform’ [24, 25, 26], which is

$$\Phi(q, p) = \frac{1}{\sqrt{2\pi}} \int \psi(u) w(u - q) e^{-ipu} du, \tag{27}$$

with $w(x)$ the window function. Choosing $w(x) = \overline{\varphi_0(-x)}$ then leads to

$$\Psi(q/2, p/2) = 2 e^{iqp/2} \Phi(q, p). \tag{28}$$

It is easily checked that Ψ and $e^{iqp/2} \Phi$ are also related by a ‘symplectic Fourier transform’,

$$e^{iqp/2} \Phi(q, p) = \frac{1}{2\pi} \int \Psi(q', p') e^{i(p'q - q'p)} d\Gamma'. \tag{29}$$

Gabor initially chose a simple Gaussian centred on the origin for the window function $\varphi_0(x)$. In the present context, this choice leads to Bargmann’s

representation [27] of the wavefunction as an entire function on Γ , regarded as the complex plane (see below).

Note also that the second formula (26) shows each transform in the general form

$$\begin{aligned}\Psi(q, p) &= \int T_{\varphi_0}(q, p, u) \psi(u) du, \\ T_{\varphi_0}(q, p, u) &= \sqrt{2/\pi} \overline{\varphi_0(2q - u)} e^{2ip(q-u)}.\end{aligned}\quad (30)$$

From (6) we can also write

$$\begin{aligned}\Psi(q, p) = \text{Tr}(|\psi\rangle\langle\varphi_0|\hat{\Delta}(q, p)) &= \langle\varphi_0|\hat{\Delta}(q, p)|\psi\rangle = \langle\mathcal{T}_{\varphi_0}(q, p)|\psi\rangle, \\ |\mathcal{T}_{\varphi_0}(q, p)\rangle &= \hat{\Delta}(q, p)|\varphi_0\rangle.\end{aligned}\quad (31)$$

In a similar way it can be seen that

$$\begin{aligned}\Phi(q, p) &= \langle\mathcal{Q}_{\varphi_0}(q, p)|\psi\rangle, \\ |\mathcal{Q}_{\varphi_0}(q, p)\rangle &= e^{i(p\hat{q}-q\hat{p})}|\varphi_0\rangle.\end{aligned}\quad (32)$$

The transform inverse to (26) is obtained by considering $\mathcal{W}^{-1}(\Psi)$ in the coordinate representation, which leads to

$$\begin{aligned}\frac{1}{2\pi} \int \Psi((x+y)/2, p) e^{ip(x-y)} dp &= \langle x|\psi\rangle\langle\varphi_0|y\rangle, \\ &= \psi(x)\overline{\varphi_0(y)}.\end{aligned}\quad (33)$$

Hence

$$\psi(x) = \frac{1}{2\pi\varphi_0(y)} \int \Psi((x+y)/2, p) e^{ip(x-y)} dp \quad (34)$$

wherever $\varphi_0(y) \neq 0$, and also

$$\psi(x) = \frac{1}{2\pi} \iint \Psi((x+y)/2, p) \varphi_0(y) e^{ip(x-y)} dy dp. \quad (35)$$

Using the definition of \mathcal{W}^{-1} in terms of $\widehat{\Delta}(q, p)$ [35], the result (35) can also be written as

$$|\psi\rangle = \frac{1}{2\pi} \int \Psi(q, p) |\mathcal{T}_{\varphi_0}(q, p)\rangle d\Gamma. \quad (36)$$

Similarly

$$|\psi\rangle = \frac{1}{2\pi} \int \Phi(q, p) |\mathcal{Q}_{\varphi_0}(q, p)\rangle d\Gamma. \quad (37)$$

4 Generalized Husimi functions

The first of formulas (26) says that $\Psi(q, p)$ is the Fourier transform of

$$A_q(y) = \psi(q - \tfrac{1}{2}y) \overline{\varphi_0(q + \tfrac{1}{2}y)}. \quad (38)$$

Similarly, $\overline{\Psi(q, p)}$ is the Fourier transform of

$$B_q(y) = \overline{A_q(-y)}, \quad (39)$$

and so $|\Psi(q, p)|^2$ is the product of these transforms. By the convolution theorem for Fourier transforms [45], it follows that

$$\begin{aligned} |\Psi(q, p)|^2 &= \frac{1}{2\pi} \int A_q(y - z) B_q(z) e^{ipy} dz dy \\ &= \frac{1}{2\pi} \int \psi(q - \tfrac{1}{2}y + \tfrac{1}{2}z) \overline{\psi(q + \tfrac{1}{2}z)} \\ &\quad \times \overline{\varphi_0(q + \tfrac{1}{2}y - \tfrac{1}{2}z)} \varphi_0(q - \tfrac{1}{2}z) e^{ipy} dz dy \\ &= \frac{2}{\pi} \int \psi(u + \tfrac{1}{2}v) \overline{\psi(u - \tfrac{1}{2}v)} \\ &\quad \times \overline{\varphi_0(2q - u - \tfrac{1}{2}v)} \varphi_0(2q - u + \tfrac{1}{2}v) e^{-2ipv} du dv. \end{aligned} \quad (40)$$

Setting

$$f(r, v) = \int \overline{\varphi_0(\theta - \frac{1}{2}v)} \varphi_0(\theta + \frac{1}{2}v) e^{ir\theta} d\theta, \quad (41)$$

so that

$$\overline{\varphi_0(\theta - \frac{1}{2}v)} \varphi_0(\theta + \frac{1}{2}v) = \frac{1}{2\pi} \int f(r, v) e^{-ir\theta} dr, \quad (42)$$

we then have

$$|\Psi(q/2, p/2)|^2 = \frac{1}{\pi^2} \int \psi(u + \frac{1}{2}v) \overline{\psi(u - \frac{1}{2}v)} f(r, v) e^{-ir(q-u)} e^{-ipv} dr du dv, \quad (43)$$

showing that $|\Psi(q/2, p/2)/2|^2$ is a (nonnegative) distribution function from Cohen's general class [25, 46]. Note from (28) that

$$|\Psi(q/2, p/2)/2|^2 d(q/2) d(p/2) = |\Phi(q, p)|^2 dq dp. \quad (44)$$

In the signals literature, $|\Phi(q, p)|^2$ is known as the spectrogram of the signal ψ . It can be regarded as a generalized Husimi function [47] (see below). In particular, when the window function is the simple Gaussian corresponding to the oscillator ground state (see below),

$$\varphi_0(x) = e^{-x^2/2}/\pi^{1/4}, \quad (45)$$

we get from (41)

$$f(r, v) = e^{-(r^2+v^2)/4}, \quad (46)$$

so that

$$|\Psi(q/2, p/2)|^2 d(q/2) d(p/2) [= |\Phi(q, p)|^2 dq dp] = Q(q, p) dq dp, \quad (47)$$

where Q is the original Husimi function [23].

It is remarkable, especially in view of (17) and (18), that $\Psi(q, p) \star \overline{\Psi(q, p)}$ equals the Wigner function for the state $|\psi\rangle$, while $\Psi(q/2, p/2) \overline{\Psi(q/2, p/2)}$ defines a generalized Husimi function for that same state, whatever the choice of window. In this way $\Psi \star \overline{\Psi}$ and $\Psi \overline{\Psi}$ define different aspects of the generalized Born interpretation of Ψ (see above).

5 Gaussian windows

Amongst the simplest choices for a window state is a squeezed state, described by a Gaussian function in the coordinate representation,

$$\begin{aligned}\varphi_0(x) &= (\beta^2/\pi)^{1/4} e^{-\beta^2(x-x_W)^2/2 + ik_W(x-x_W/2)} \\ &= (\beta^2/\pi)^{1/4} e^{\beta^2\lambda(\lambda-\bar{\lambda})/4} e^{-\beta^2(x-\lambda)^2/2}, \quad \lambda = x_W + ik_W/\beta^2,\end{aligned}\quad (48)$$

where β , x_W and k_W are real constants. This has momentum space representation (Fourier transform)

$$\begin{aligned}\tilde{\varphi}_0(k) &= (1/\beta^2\pi)^{1/4} e^{-(k-k_W)^2/2\beta^2 - i(k-k_W/2)x_W} \\ &= (1/\beta^2\pi)^{1/4} e^{-\beta^2\lambda(\lambda+\bar{\lambda})/4} e^{-(k+i\beta^2\lambda)^2/2\beta^2}.\end{aligned}\quad (49)$$

Then

$$|\varphi_0(x)|^2 = e^{-\beta^2(x-x_W)^2}, \quad |\tilde{\varphi}_0(k)|^2 = e^{-(k-k_W)^2/\beta^2}, \quad (50)$$

which are peaked at $x = x_W$ and $k = k_W$ respectively. Note also that $\varphi_0(x)$ is the coordinate space representation of a solution of

$$\frac{1}{\sqrt{2}}(\beta\hat{q} + i\hat{p}/\beta)|\varphi_0\rangle = \frac{\beta\lambda}{\sqrt{2}}|\varphi_0\rangle. \quad (51)$$

Before proceeding, we recall that

$$\mathcal{W}(\hat{q}^n) = q \star q \star q \cdots \star q = q^n, \quad \mathcal{W}(\hat{p}^n) = p^n, \quad (52)$$

and that, following Bopp [48, 40]

$$\begin{aligned}\mathcal{W}(\hat{A}\hat{q}^n)(q, p) &= A(q, p) \star q^n = [q - \tfrac{1}{2}i\partial_p]^n A(q, p) \equiv [q_{BR}]^n A(q, p), \\ \mathcal{W}(\hat{A}\hat{p}^n)(q, p) &= A(q, p) \star p^n = [p + \tfrac{1}{2}i\partial_q]^n A(q, p) \equiv [p_{BR}]^n A(q, p), \\ \mathcal{W}(\hat{q}^n\hat{A})(q, p) &= q^n \star A(q, p) = [q + \tfrac{1}{2}i\partial_p]^n A(q, p) \equiv [q_{BL}]^n A(q, p), \\ \mathcal{W}(\hat{p}^n\hat{A})(q, p) &= p^n \star A(q, p) = [p - \tfrac{1}{2}i\partial_q]^n A(q, p) \equiv [p_{BL}]^n A(q, p).\end{aligned}\quad (53)$$

Applying the first two of these results to $\widehat{A} = |\psi\rangle\langle\varphi_0|$ with $n = 1$, it follows from (51) that every phase space amplitude Ψ constructed with the Gaussian window state (48) must satisfy

$$\frac{1}{\sqrt{2}}(\beta q_{BR} - ip_{BR}/\beta)\Psi(q, p) = \frac{\beta\bar{\lambda}}{\sqrt{2}}\Psi(q, p), \quad (54)$$

and it is easily seen that the general solution of this equation has the form

$$\Psi(q, p) = \sqrt{2/\pi} e^{-\bar{z}z/2 + \beta(z+\bar{z})\bar{\lambda}/\sqrt{2}} G_\psi(z), \quad z = \sqrt{2}(\beta q - ip/\beta), \quad (55)$$

where G_ψ is arbitrary and we have included an extra z -dependent prefactor to G_ψ for later convenience. The form of $G_\psi(z)$ is determined by ψ from (48) and the second formula in (26), leading to

$$G_\psi(z) = (\beta^2/\pi)^{1/4} e^{\beta^2\bar{\lambda}(\bar{\lambda}-\lambda)/4} \times \int e^{-(z^2 + \beta^2(u+\bar{\lambda})^2 - 2\sqrt{2}\beta zu)/2} \psi(u) du. \quad (56)$$

6 Generalized Bargmann representation

Because the exponential factor in (55) is independent of ψ , this result establishes a generalized Bargmann representation [27],

$$|\psi\rangle \longleftrightarrow G_\psi(z), \quad (57)$$

with G_ψ an analytic function of the complex variable z (no \bar{z} -dependence), or equivalently of $p + i\beta^2 q$, in a Hilbert space \mathcal{H}_G with weighted scalar product, from (14) and (55),

$$(G_{\psi_1}, G_{\psi_2}) = (2/\pi) \int \overline{G_{\psi_1}(z)} G_{\psi_2}(z) e^{-\bar{z}z + \beta(\lambda + \bar{\lambda})(z + \bar{z})/\sqrt{2}} d^2 z. \quad (58)$$

Here $d^2 z = dq dp$. The creation and annihilation operators $(\widehat{q} \mp i\widehat{p})/\sqrt{2}$ acting on each $|\psi\rangle$ in \mathcal{H} , are represented on \mathcal{S}_{φ_0} as

$$(q_{BL} \mp ip_{BL})/\sqrt{2} \equiv \pm \frac{1}{2}(\beta \pm 1/\beta)(\frac{1}{2}z - \partial_{\bar{z}}) \mp \frac{1}{2}(\beta \mp 1/\beta)(\frac{1}{2}\bar{z} + \partial_z), \quad (59)$$

and therefore, taking into account the exponential factor in (55), they are represented on \mathcal{H}_G as

$$\begin{aligned} (\hat{q} - i\hat{p})/\sqrt{2} &\longleftrightarrow \sigma z - \tau \partial_z - \beta^2 \bar{\lambda}/\sqrt{2} \\ (\hat{q} + i\hat{p})/\sqrt{2} &\longleftrightarrow \sigma \partial_z - \tau z + \beta^2 \bar{\lambda}/\sqrt{2}, \\ \sigma &= (\beta^2 + 1)/2\beta, \quad \tau = (\beta^2 - 1)/2\beta. \end{aligned} \quad (60)$$

Then

$$\begin{aligned} \hat{q} &\longleftrightarrow (z + \partial_z)/\beta\sqrt{2} \\ \hat{p} &\longleftrightarrow i\beta(z - \partial_z)/\sqrt{2} - i\beta^2 \bar{\lambda}. \end{aligned} \quad (61)$$

Note that the RHSs of (60) and (61) do not involve \bar{z} and so preserve the analyticity of any G_ψ upon which they act. They satisfy the canonical commutation relations and have the appropriate hermiticity properties with respect to the scalar product (58). These expressions generalize those of the usual Bargmann representation, which corresponds to the case $\beta = 1$ and $\lambda = 0$, that is, the case of a Gaussian window (48) with the same scaling as that used for \hat{q} and \hat{p} , and centered on $x = 0$. Then $\sigma = 1$ and $\tau = 0$, and the expressions on the RHS of (60) reduce to the familiar z and ∂_z , respectively. Furthermore, it can be seen that (56) reduces when $\beta = 1$ and $\lambda = 0$ to Definition (2.3) of the Bargmann transform in [27].

The intimate connection between the usual Bargmann wavefunction and the Wigner function, and with the phase space formulation of quantum mechanics more generally, has been discussed previously [49, 50, 51] from various other points of view. Furthermore, it has recently been shown [44] in the context of the Gabor transform, that only Gaussian windows give rise to spaces of analytic functions of the (generalized) Bargmann type.

7 A test state and its phase space amplitude

Consider a normalized test wavefunction

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{\sqrt{\pi}(1+2\sqrt{2})}} \left(e^{-(x-1)^2/2} + 4ix e^{-x^2} \right) \\ \longleftrightarrow \\ \tilde{\psi}(k) &= \frac{1}{\sqrt{\sqrt{\pi}(1+2\sqrt{2})}} \left(e^{-k^2/2-ik} + \sqrt{2}k e^{-k^2/4} \right). \end{aligned} \quad (62)$$

which has as Wigner function

$$\begin{aligned} W(q, p) &= \frac{1}{\pi(1+2\sqrt{2})} \left\{ e^{-(q-1)^2-p^2} \right. \\ &+ 2\sqrt{2}(4q^2 + p^2 - 1) e^{-2q^2-p^2/2} - \frac{8\sqrt{2}}{3\sqrt{3}} [(2q-1) \sin(2p[q+1]/3) \\ &\left. - 2p \cos(2p[q+1]/3)] e^{-(4q^2-4q+2p^2+1)/3} \right\} \end{aligned} \quad (63)$$

and which, for the choice (48) as window with $x_W = \langle \hat{q} \rangle$ and $k_W = \langle \hat{p} \rangle$, leads to the phase space amplitude

$$\begin{aligned} \Psi(q, p) &= N \left\{ e^{-[4\beta^2 q^2 + 4p^2 + 4i(\beta^2-1)qp - 4\beta^2(\bar{\lambda}+1)q]/2(\beta^2+1)} \right. \\ &\times e^{[4i(1-\beta^2\bar{\lambda})p + \beta^2(\bar{\lambda}+1)^2]/2(\beta^2+1)} / (\beta^2+1)^{1/2} \\ &+ 4i(2\beta^2 q - 2ip - \beta^2\bar{\lambda}) e^{-[4\beta^2 q^2 + 2p^2 + 2i(\beta^2-2)qp]/(\beta^2+2)} \\ &\left. \times e^{[4\beta^2\bar{\lambda}q + 2i\beta^2\bar{\lambda}p - \beta^2\bar{\lambda}^2]/(\beta^2+2)} / (\beta^2+2)^{3/2} \right\}. \end{aligned} \quad (64)$$

Here

$$\begin{aligned} N &= \sqrt{4\beta/\pi(1+2\sqrt{2})} e^{\beta^2\bar{\lambda}(\bar{\lambda}-\lambda)/4}, \quad \lambda = x_W + ik_W/\beta^2, \\ x_W = \langle \hat{q} \rangle &= 1/(1+2\sqrt{2}), \quad k_W = \langle \hat{p} \rangle = 8\sqrt{2}/9 e^{1/3}\sqrt{3}(1+2\sqrt{2}). \end{aligned} \quad (65)$$

The values of x_W and k_W in the definition (48) of the window function have been equated to the expectation values $\langle \hat{q} \rangle$, $\langle \hat{p} \rangle$ of position and momentum for the state (62), so that $|\varphi_0(x)|$ and $|\tilde{\varphi}_0(k)|$ are localized near these values in position and momentum space, respectively, in accordance with (50).

Fig. 1 shows the real and imaginary parts of Ψ as in (64) when $\beta = 1$. Fig. 2 shows the Wigner function W of (63), which is not everywhere positive, and the distribution $|\Psi|^2$ in this case, for Ψ as in (64), with $\beta = 1$. Fig. 3 shows $|\Psi|^2$ for $\beta = 0.5$ and 2. Comparison of the subplots shows the influence of the uncertainty principle: when $\beta^2 \ll 1$, the structure of $|\Psi|^2$ better delineates the p -dependence of W , while when $\beta^2 \gg 1$, it better delineates the q -dependence.

[Figs. 1, 2 and 3 near here.]

8 Coherent states in phase space

The formulas (31) and (32) show that Ψ and Φ can always be regarded as generalized coherent states in Perelemov's sense [52]. However, we now ask what phase space amplitudes Ψ correspond to the familiar coherent state in \mathcal{H} defined up to a constant phase by

$$\frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})|\psi_\mu\rangle = \frac{\mu}{\sqrt{2}}|\psi_\mu\rangle, \quad \mu = x_C + ik_C \in \mathbb{C} \quad (66)$$

and normalization. Here x_C and k_C are arbitrary real numbers. It follows from (53) that each such state has a corresponding phase space amplitude Ψ_μ satisfying

$$\begin{aligned} \frac{1}{\sqrt{2}}(q_{BL} + ip_{BL})\Psi_\mu(q, p) &\equiv \frac{1}{\sqrt{2}}(q + \tfrac{1}{2}i\partial_p + ip + \tfrac{1}{2}\partial_q)\Psi_\mu(q, p) \\ &= \frac{\mu}{\sqrt{2}}\Psi_\mu(q, p), \end{aligned} \quad (67)$$

and hence having the general form

$$\Psi_\mu(q, p) = K_\mu(\bar{w}) e^{-\bar{w}w/2} e^{\mu w/\sqrt{2}}, \quad w = \sqrt{2}(q - ip). \quad (68)$$

Here the precise form of K_μ is determined by the phase and normalization of $|\psi_\mu\rangle$ and the choice of window $|\varphi_0\rangle$. Each choice of $|\varphi_0\rangle$ leads also to a corresponding subspace \mathcal{S}_{φ_0} of square integrable functions on Γ through (22) and (23), and it is remarkable that the phase space amplitudes (68) — which we may call coherent states in phase space — must, for every choice of window state, form an overcomplete set in the corresponding \mathcal{S}_{φ_0} for varying μ , just as the coherent states $|\psi_\mu\rangle$ form an overcomplete set in \mathcal{H} .

If the window is a Gaussian as in (48), then the amplitude Ψ_μ must also satisfy (54), leading from (68) to

$$K_\mu(\bar{w}) = \text{const.} e^{-\tau\bar{w}^2/2\sigma} e^{(\tau\mu+\beta\bar{\lambda})\bar{w}/\sqrt{2}\sigma}, \quad (69)$$

where σ and τ are as in (60). In this case we know also that Ψ_μ must be of the form (55), and we find

$$\begin{aligned} \Psi_\mu(q, p) &= V_\mu(z) e^{-\bar{z}z/2} e^{\beta\bar{\lambda}\bar{z}/\sqrt{2}}, \quad z = \sqrt{2}(\beta q - ip/\beta), \\ V_\mu(z) &= \text{const.} e^{\tau z^2/2\sigma} e^{(\mu-\tau\beta\bar{\lambda})z/\sigma\sqrt{2}}. \end{aligned} \quad (70)$$

Consistency of (68), (69) and (70) is easily checked, leading to

$$\Psi_\mu(q, p) = \text{const.} e^{-2[\beta^2 q^2 + p^2 + i(\beta^2 - 1)qp - \beta^2(\bar{\lambda} + \mu)q - i(\beta^2\bar{\lambda} - \mu)p]/(\beta^2 + 1)}, \quad (71)$$

and hence

$$\begin{aligned} \overline{\Psi_\mu(q, p)} \Psi_\mu(q, p) &= \frac{4\beta}{\pi(\beta^2 + 1)} e^{-4[\beta^2(q-\eta)^2 + (p-\zeta)^2]/(\beta^2 + 1)}, \\ \eta &= (x_W + x_C)/2, \quad \zeta = (k_W + k_C)/2, \end{aligned} \quad (72)$$

with x_W, k_W as in (48) and x_C, k_C as in (66). In (72), we have taken the normalization condition (15) into account. On the other hand, the Wigner function corresponding to the coherent state $|\psi_\mu\rangle$ is [22]

$$W_\mu(q, p) = \frac{1}{\pi} e^{-(q-x_C)^2 - (p-k_C)^2}. \quad (73)$$

If we choose the Gaussian window to be centered on the same coordinate and momentum values x_C, k_C as the Wigner function, with the same choice of length scale — that is to say, if we choose $\lambda = \mu$ and $\beta = 1$ — then

$$\overline{\Psi_\mu(q, p)} \Psi_\mu(q, p) = \frac{2}{\pi} e^{-2(q-x_C)^2 - 2(p-k_C)^2} \quad (74)$$

which is also centered on the same values as the Wigner function. Note however that these two distributions (73) and (74) are not equal, even in this case when both are positive.

Comparing (72) with (73) and (74), we see also the effect of choosing a window that is *not* centred on the key features of the Wigner function.

9 Schrödinger's equation in phase space and evolution of amplitudes

From (2), corresponding to an evolving state vector $|\psi(t)\rangle$ in \mathcal{H} , we have for a time-dependent amplitude in phase space, Schrödinger's equation in the form

$$i \partial_t \Psi(q, p, t) = H(q, p) \star \Psi(q, p, t), \quad (75)$$

where $H = \mathcal{W}(\hat{H})$. Supposing H is not explicitly time-dependent, this equation integrates to give

$$\Psi(q, p, t) = U(t - t_0) \star \Psi(q, p, t_0), \quad U(t) = e^{\star[-iH(q,p)t]}, \quad (76)$$

where the star exponential is defined formally by

$$e^{\star[A]} = \mathcal{W}(e^{\hat{A}}) = 1 + A + A \star A/2! + A \star A \star A/3! + \dots \quad (77)$$

When H is a polynomial in q and p , a more explicit form for the time-evolution may be available. For example, if H describes a non-relativistic particle and has the form

$$H(q, p) = \frac{1}{2}p^2 + V(q), \quad (78)$$

with V a polynomial in q , then from (53) we can rewrite (75) as

$$i \partial_t \Psi(q, p, t) = H(q_{BL}, p_{BL}) \Psi(q, p, t), \quad (79)$$

leading to

$$\Psi(q, p, t) = e^{-iH(q_{BL}, p_{BL})(t-t_0)} \Psi(q, p, t_0). \quad (80)$$

An efficient way to solve (75) or (79) explicitly, when this is possible, is to solve Schrödinger's equation in the coordinate representation to get the wavefunction $\psi(x, t)$ and then use that in (26) to construct $\Psi(q, p, t)$.

For example, consider the initial-value problem for a free particle

$$i \partial_t \psi(x, t) = -\frac{1}{2} \partial^2 \psi(x, t) / \partial x^2, \quad \psi(x, 0) = \sqrt{\gamma} e^{-\gamma^2 x^2 / 2} / \pi^{1/4} \quad (81)$$

where γ is a positive constant, with solution

$$\psi(x, t) = \sqrt{\gamma} e^{-\gamma^2 x^2 / 2g(t)^2} / g(t) \pi^{1/4}, \quad g(t) = [1 + i\gamma^2 t]^{1/2}, \quad t \geq 0. \quad (82)$$

Here the branch of the complex square-root is chosen so that $g(0) = 1$. With the Gaussian (48) as window, (82) gives the corresponding time-dependent phase space amplitude

$$\begin{aligned} \Psi(q, p, t) &= [4\beta\gamma/\pi(\beta^2 g(t)^2 + \gamma^2)]^{1/2} e^{\beta^2 \bar{\lambda}(\bar{\lambda}-\lambda)/4} e^{-\beta^2 \bar{\lambda}^2 \gamma^2 / 2(\beta^2 g(t)^2 + \gamma^2)} \\ &\times e^{-2[\beta^2 \gamma^2 q^2 + p^2 g(t)^2 + i(\beta^2 g(t)^2 - \gamma^2)qp - \beta^2 \gamma^2 \bar{\lambda}q - i\beta^2 \bar{\lambda}pg(t)^2]/(\beta^2 g(t)^2 + \gamma^2)}, \end{aligned} \quad (83)$$

as the solution of

$$i \partial_t \Psi(q, p, t) = \frac{1}{2} p_{BL}^2 \Psi(q, p, t), \quad (84)$$

corresponding to the initial value

$$\begin{aligned} \Psi(q, p, 0) &= [4\beta\gamma/\pi(\beta^2 + \gamma^2)]^{1/2} e^{\beta^2 \bar{\lambda}(\bar{\lambda}-\lambda)/4} e^{-\beta^2 \bar{\lambda}^2 \gamma^2 / 2(\beta^2 + \gamma^2)} \\ &\times e^{-2[\beta^2 \gamma^2 q^2 + p^2 + i(\beta^2 - \gamma^2)qp - \beta^2 \gamma^2 \bar{\lambda}q - i\beta^2 \bar{\lambda}p]/(\beta^2 + \gamma^2)}, \end{aligned} \quad (85)$$

which may be compared with (71) in the case $\gamma = 1, \mu = 0$.

If the eigenvalue problem

$$H(q, p) \star \Psi(q, p) = E \Psi(q, p) \quad (86)$$

can be solved to find a complete set of phase space eigenfunctions $\Psi_{E_n}(q, p)$ — some will be generalized eigenfunctions if \hat{H} has a (partly) continuous spectrum — then a more explicit form of solution to (75) is

$$\begin{aligned} \Psi(q, p, t) &= \sum_n c_n e^{-iE_n(t-t_0)} \Psi_{E_n}(q, p), \\ \Psi(q, p, t_0) &= \sum_n c_n \Psi_{E_n}(q, p). \end{aligned} \quad (87)$$

Here each sum must be extended to include an integral over the continuous spectrum, when appropriate. The eigenvalue problem (86) is distinct from the ‘ \star -genvalue’ problem discussed in the context of the (one-state) Wigner function [53, 54], which can be expressed in terms of the solution of (86) using (16).

The expressions (87) are the images under \mathcal{W} of the coordinate space formulas

$$\begin{aligned} \psi(x, t) &= \sum_n c_n e^{-iE_n(t-t_0)} \psi_n(x), \quad \psi(x, t_0) = \sum_n c_n \psi_n(x), \\ \text{where } \hat{H}\psi_n &= E_n \psi_n. \end{aligned} \quad (88)$$

Similarly, orthogonality of the coordinate space eigenfunctions and determination of the expansion coefficients,

$$\int \bar{\psi}_m(x) \psi_n(x) dx = \delta_{mn}, \quad c_n = \int \bar{\psi}_n(x) \psi(x, t_0) dx \quad (89)$$

have images

$$\int \bar{\Psi}_m(q, p) \Psi_n(q, p) d\Gamma = \delta_{mn}, \quad c_n = \int \bar{\Psi}_n(q, p) \Psi(q, p, t_0) d\Gamma. \quad (90)$$

If the ψ_n form a complete orthonormal set in the coordinate space representation of Hilbert space, so the Ψ_n form a complete orthonormal set in \mathcal{S}_{φ_0} .

The simple harmonic oscillator with $V(q) = \frac{1}{2}q^2$ provides the simplest illustration. In this case, (10) shows that (86) becomes

$$[\frac{1}{2}(q^2 + p^2) + \frac{1}{2}i(q\partial_p - p\partial_q) - \frac{1}{8}(\partial_q^2 + \partial_p^2)]\Psi(q, p) = E\Psi(q, p) \quad (91)$$

which is easily solved using a phase space variant of the boson calculus. Set

$$\begin{aligned} \check{A} &= [(q + \frac{1}{2}i\partial_p) + i(p - \frac{1}{2}i\partial_q)]/\sqrt{2} = \frac{1}{2}\bar{w} + \partial_w, \\ \check{A}^\dagger &= [(q + \frac{1}{2}i\partial_p) - i(p - \frac{1}{2}i\partial_q)]/\sqrt{2} = \frac{1}{2}w - \partial_{\bar{w}}, \end{aligned} \quad (92)$$

where we have used checks to distinguish phase space operators, and again introduced $w = \sqrt{2}(q - ip)$ as in (68).

Solving $\check{A}\Psi_0 = 0$ for the phase space ‘vacuum state’, normalizing it, and then setting $\Psi_n = \check{A}^{\dagger n}\Psi_0/\sqrt{n!}$ for $n = 1, 2, \dots$, we find

$$\Psi_n(q, p) = \frac{1}{\sqrt{n!}} \sum_{m=0}^n C_m^n (-1)^{n-m} w^m F^{(n-m)}(\bar{w}) e^{-\bar{w}w/2}, \quad (93)$$

corresponding to the familiar eigenvalue $E_n = n + \frac{1}{2}$, for $n = 0, 1, 2, \dots$. In (93), $F(\bar{w})$ and its derivatives are determined by the choice of window and the normalization of Ψ_0 , which requires

$$\int \overline{F(\bar{w})} F(\bar{w}) e^{-\bar{w}w} d^2w = 1. \quad (94)$$

The obvious choice in the present context, $\varphi_0(x)$ as in (48) with $\beta = 1$, $\lambda = 0$, leads to $F = \sqrt{2/\pi}$, and (93) is then the Bargmann wavefunction for the n -th oscillator eigenstate, apart from the exponential factor. But again we emphasize that any convenient window function can be chosen, and every choice leads through (16) to the same Wigner functions, which in this case are [22]

$$W_n(q, p) = (-1)^n L_n(2[q^2 + p^2]) e^{-(q^2+p^2)}/\pi, \quad (95)$$

where L_n is the Laguerre polynomial [55]. Different choices of φ_0 lead to different phase space amplitudes Ψ_n and hence to different distributions $|\Psi_n|^2$

but the same Wigner function (95). For example, in the case $n = 1$, the choice (48) with $\beta = 1$, $\lambda = 0$ gives from (93)

$$\Psi_1(q, p) = (\sqrt{2/\pi}) w e^{-\bar{w}w/2} \quad (96)$$

and hence

$$|\Psi_1(q, p)|^2 = (2/\pi) (\bar{w}w) e^{-\bar{w}w}, \quad (97)$$

whereas the choice of a ‘square’ window

$$\varphi_0(x) = \begin{cases} 1/\sqrt{2a} & : |x| < a \\ 0 & : \text{otherwise} \end{cases} \quad (98)$$

leads to

$$F(\bar{w}) = (1/\pi^{1/4} \sqrt{a}) e^{\bar{w}^2/2} \left[\text{erf}(a/\sqrt{2} - \bar{w}) + \text{erf}(a/\sqrt{2} + \bar{w}) \right] \quad (99)$$

in (93) and from there to

$$\begin{aligned} \Psi_1(q, p) = & \\ & (1/\pi^{1/4} \sqrt{2a}) e^{-\bar{w}w/2} \left\{ (w - \bar{w}) e^{\bar{w}^2/2} \left[\text{erf}(a/\sqrt{2} - \bar{w}) + \text{erf}(a/\sqrt{2} + \bar{w}) \right] \right. \\ & \left. + 4/\sqrt{\pi} e^{-(a^2 + \bar{w}^2)/2} \sinh(\sqrt{2}a\bar{w}) \right\}. \end{aligned} \quad (100)$$

Fig. 4 shows the Wigner function W_1 of (95), and $|\Psi_1|^2$ for the Gaussian window with $\beta = 1$, $\lambda = 0$ as in (97). Fig. 5 shows $|\Psi_1|^2$ for Ψ_1 as in (100) in the case of a square window with $a = 1$.

[Figs. 4 and 5 near here.]

10 Eigenstates of momentum and position in phase space

In view of (53), a (generalized) eigenstate of momentum in phase space is defined by

$$p_{BL} \Psi_{k_0}(q, p) = k_0 \Psi_{k_0}(q, p), \quad (101)$$

giving

$$\Psi_{k_0}(q, p) = F_{k_0}(p) e^{-2i(p-k_0)q}, \quad (102)$$

with $F_{k_0}(p)$ undetermined. The form of F_{k_0} depends on the choice of $|\varphi_0\rangle$. In fact, when the generalized momentum eigenfunction (plane wave)

$$\psi(x) = e^{ik_0 x} / \sqrt{2\pi} \quad (103)$$

is inserted into (26), it is revealed that

$$\Psi_{k_0}(q, p) = \sqrt{2/\pi} \overline{\tilde{\varphi}_0(2p - k_0)} e^{-2i(p-k_0)q}, \quad (104)$$

where $\tilde{\varphi}_0(k) = \langle k | \varphi_0 \rangle$ is the function in momentum space that corresponds to (= Fourier transform of) $\varphi_0(x)$. It can be checked directly using the integral form of the star product [36, 37, 38, 41],

$$\begin{aligned} & (A \star B)(q, p) \\ &= \frac{1}{\pi^2} \int_{\Gamma_2 \times \Gamma_3} A(q + q_2, p + p_2) B(q + q_3, p + p_3) e^{2i(q_2 p_3 - q_3 p_2)} \\ & \quad \times d\Gamma_2 d\Gamma_3 \end{aligned} \quad (105)$$

that (16) does indeed hold in this case whatever choice is made for $|\varphi_0\rangle$, in every case yielding the (singular) Wigner function

$$W_{k_0}(q, p) = \delta(p - k_0) / 2\pi. \quad (106)$$

In a similar way, the generalized phase space position eigenfunction is defined by

$$q_{BL} \Psi_{x_0}(q, p) = x_0 \Psi_{x_0}(q, p) \quad (107)$$

and we find

$$\Psi_{x_0}(q, p) = \sqrt{2/\pi} \overline{\varphi_0(2q - x_0)} e^{2ip(q - x_0)}, \quad (108)$$

corresponding to $W_{x_0}(q, p) = \delta(q - x_0)/2\pi$. Noting that Ψ_{x_0} in (108) is $\mathcal{W}(|x_0\rangle\langle\varphi_0|)/\sqrt{2\pi}$, we then see that the second of formulas (26) is simply the image under the Weyl-Wigner transform of the formula

$$|\psi\rangle\langle\varphi_0| = \int |u\rangle\langle u|\psi\rangle\langle\varphi_0| du = \int \psi(u) |u\rangle\langle\varphi_0| du. \quad (109)$$

Formulas (104) and (108) show again the benefit of choosing a window state that is centered on a region of interest in phase space. For example if we choose a Gaussian window centered on coordinate and momentum values x_W and k_W as in (48), (49), (50), then for the phase space momentum eigenfunction (104) we get

$$\begin{aligned} \Psi_{k_0}(q, p) &= (4/\pi^3\beta^2)^{1/4} e^{-(2p - k_0 - k_W)^2/2\beta^2} \\ &\times e^{-i(2qp - 2qk_0 - 2x_W p + x_W k_0 + x_W k_W/2)}. \end{aligned} \quad (110)$$

If the window state has $k_W = k_0$ and so is centred on the momentum value k_0 of interest, then

$$|\Psi_{k_0}(q, p)|^2 = (4/\pi^3\beta^2)^{1/2} e^{-4(p - k_0)^2/\beta^2}, \quad (111)$$

which is also centred on $p = k_0$ for each value of q , and may be compared with the Wigner function (106).

Similarly, for the phase space position eigenstate (108) with a Gaussian window we get

$$\begin{aligned} \Psi_{x_0}(q, p) &= (4\beta^2/\pi^3)^{1/4} e^{-\beta^2(2q - x_0 - x_W)^2/2} \\ &\times e^{i(2qp - 2x_0 p - 2qk_W + x_0 k_W + x_W k_W/2)}, \end{aligned} \quad (112)$$

so that if the window state has $x_W = x_0$, then

$$|\Psi_{x_0}(q, p)|^2 = (2\beta^2/\pi^3)^{1/2} e^{-2\beta^2(q-x_0)^2}, \quad (113)$$

centered on $q = x_0$ for each value of p . Fig. 6 shows the real part of Ψ_{k_0} for $k_0 = -2$, with the parameter choices $k_W = k_0$, $x_W = 4$ and $\beta = 1$ in the Gaussian window (48). The imaginary part is similar in form.

[Fig. 6 near here.]

11 Oscillator states as windows

An obvious generalization of the Gaussian window state with wavefunction $\varphi_0(x)$ as in (48), is the n -th excited state of the oscillator, for some non-negative integer n , more precisely, the state obtained by applying a suitable ‘creation operator’ n times to the Gaussian ‘ground state’, to give the normalized wavefunction

$$\check{\varphi}_0(x) = (1/\sqrt{n!}) [(\beta(x - \bar{\lambda}) - \partial_x/\beta)/\sqrt{2}]^n \varphi_0(x). \quad (114)$$

Windows of this type have been considered recently [56] in the context of the Gabor transform; just as the Gaussian choice leads to the (generalized) Bargmann transform as noted above, so these oscillator windows lead to further generalizations of the Bargmann transform and Bargmann representation.

It follows at once from (114) that if $\Psi(q, p)$ is the amplitude corresponding to a system state $|\psi\rangle$ and window state $|\varphi_0\rangle$, then the amplitude corresponding to the state $|\psi\rangle$ and window state $|\check{\varphi}_0\rangle$ is

$$\check{\Psi}(q, p) = (1/\sqrt{n!}) [(\beta(q_{BR} - \lambda) + ip_{BR}/\beta)/\sqrt{2}]^n \Psi(q, p), \quad (115)$$

with q_{BR} , p_{BR} as in (53).

For example, the coherent state ψ_μ defined by (66) leads to the amplitude Ψ_μ as in (71) when the Gaussian window (48) is chosen, whereas the choice of the $n = 1$ (‘Mexican Hat’) excited state in (114) as window leads through (115) to the amplitude

$$\check{\Psi}_\mu(q, p) =$$

$$\sqrt{2}\beta[(2q - x_W - x_C) + i(2p - k_W - k_C)] \Psi_\mu(q, p)/(\beta^2 + 1), \quad (116)$$

and hence from (72) to the non-negative expression

$$\begin{aligned} & \overline{\check{\Psi}_\mu(q, p)} \check{\Psi}_\mu(q, p) \\ &= \frac{32\beta^3}{\pi(\beta^2 + 1)^3} [(q - \eta)^2 + (p - \zeta)^2] e^{-4[\beta^2(q - \eta)^2 + (p - \zeta)^2]/(\beta^2 + 1)}, \quad (117) \end{aligned}$$

with η, ζ as in (72). This expression should be compared with $|\Psi_\mu(q, p)|^2$ as in (72), and with $W_\mu(q, p)$ as in (73).

12 Concluding remarks

The introduction of phase space amplitudes extends the phase space formulation of quantum mechanics, and may be considered to complete that formulation by providing images of not only density operators but also state vectors.

The degree of arbitrariness in the definition of the amplitude chosen to represent a given quantum state vector reflects the freedom to choose any normalized state as window state, and may seem surprising at first, notwithstanding our final remarks in Section 2. Further study is needed to see how to optimize the choice of window for a given quantum system in a given state, for purposes of computation and visualization. Selecting a Gaussian window with adjustable width and location in coordinate space and momentum space is simple and natural, and often leads to analytic expressions for amplitudes representing simple quantum states, as we have seen. This has been useful for our purpose here, which was to illustrate basic ideas, but it may not always be optimal. A natural choice of window for an evolving quantum system with a well-defined ground state and excited states might be its ground state, for example.

Some of the analysis of amplitudes corresponding to quantum states at a given instant, mirrors studies of signals in the time-frequency domain done

over many years using Gabor's windowed Fourier transform [26, 44, 56]. A feature of the situation in quantum mechanics that is absent in the case of signal processing arises from the very different role that the time variable plays in the two cases. In the quantum case, the time is not one of the phase space variables, and the description of quantum phase space amplitudes evolving in time is a feature absent in the case of signals. Given that the window state once chosen is fixed, and does not evolve in time, the choice of an optimal window is therefore a more complicated problem in the quantum case.

Other questions that seem worthy of further study in terms of phase space amplitudes include the formulation of the uncertainty principle, the description of interference effects, and the description of quantum symmetries. We hope to return to some of these questions.

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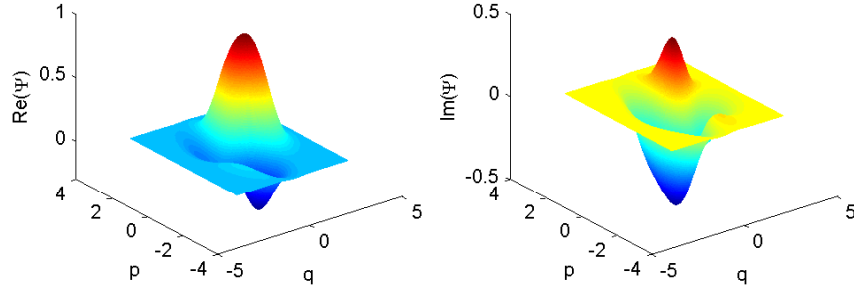


Figure 1: The real and imaginary parts of the phase space amplitude (64) with $\beta = 1$.

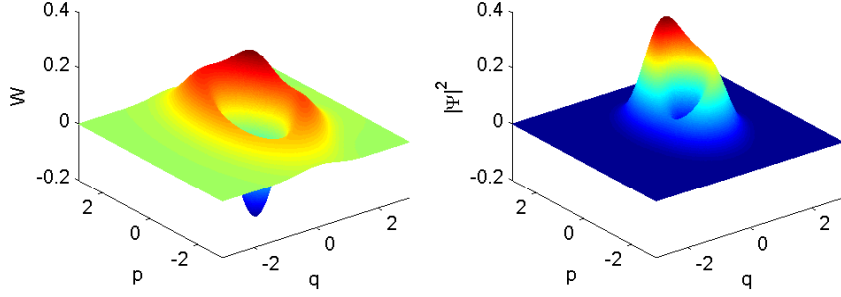


Figure 2: Left to right: the Wigner function (63) and $|\Psi|^2$, for Ψ as in (64) and $\beta = 1$.

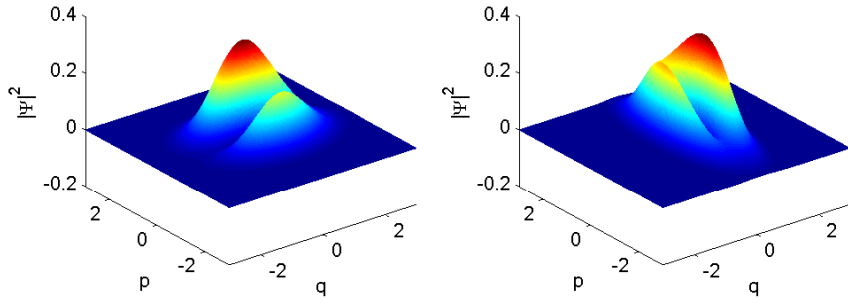


Figure 3: Left to right: $|\Psi|^2$, for Ψ as in (64) and $\beta = 0.5, 2$.

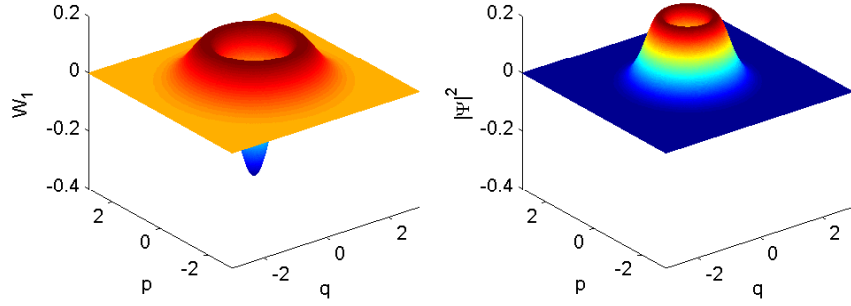


Figure 4: The Wigner function W_1 of (95) and $|\Psi_1|^2$ as in (97), for a Gaussian window having $\beta = 1$ and $\lambda = 0$.

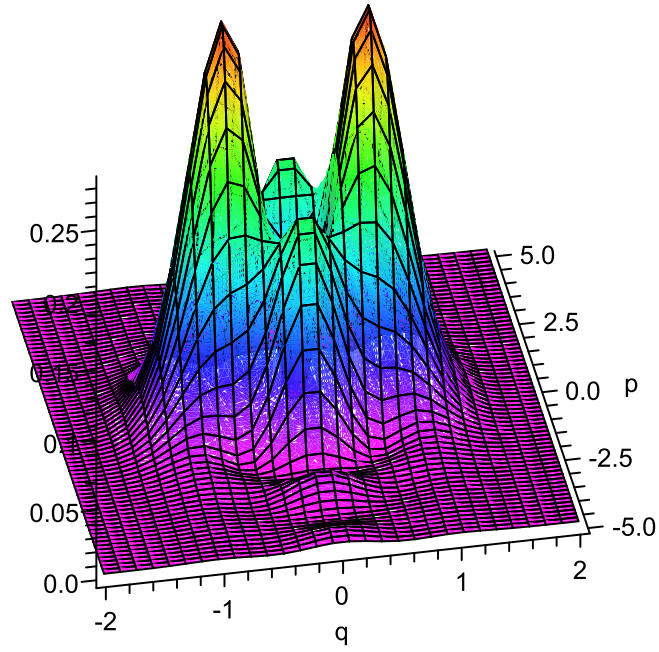


Figure 5: The quantity $|\Psi_1|^2$ for Ψ_1 as in (100), for a square window having $a = 1$.

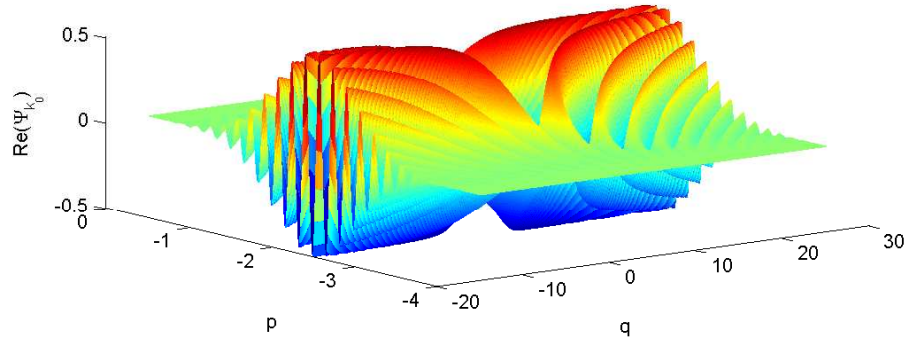


Figure 6: $\text{Re}(\Psi_{k_0=-2})$ for the amplitude Ψ_{k_0} as in (110), with a Gaussian window having $x_W = 4$, $k_W = -2$ and $\beta = 1$.